

# Kirszbraun extension on connected finite graph

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November 6, 2015

**Abstract:** We prove that the tight function introduced Sheffield and Smart (2012) [10] is a Kirszbraun extension. In the real-valued case we prove that Kirszbraun extension is unique. Moreover, we produce a simple algorithm which calculates efficiently the value of Kirszbraun extension in polynomial time.

**Key words:** Minimal, Lipschitz, extension, Kirszbraun, harmonious.

## 1 Introduction

Let  $A$  be a compact subset of  $\mathbb{R}^n$ . The best Lipschitz constant of a Lipschitz function  $g : A \rightarrow \mathbb{R}^m$  is

$$\text{Lip}(g, A) := \sup_{x \neq y \in A} \frac{\|g(x) - g(y)\|}{\|x - y\|}, \quad (1)$$

where  $\|\cdot\|$  is Euclidean norm.

When  $m = 1$ , Aronsson in 1967 [1] proved the existence of absolutely minimizing Lipschitz extension (AMLE), i.e., a extension  $u$  of  $g$  satisfying

$$\text{Lip}(u; V) = \text{Lip}(u, \partial V), \quad \text{for all } V \subset \subset \mathbb{R}^n \setminus A. \quad (2)$$

Jensen in 1993 [5] proved the uniqueness of AMLE under certain conditions.

In this chapter we begin by studying the discrete version of the existence and uniqueness of AMLE for case  $m \geq 2$ .

We define the function

$$\lambda(g, A)(x) := \inf_{y \in \mathbb{R}^m} \sup_{a \in A} \frac{\|g(a) - y\|}{\|a - x\|} \quad \text{if } x \in \mathbb{R}^n \setminus A. \quad (3)$$

From Kirszbraun theorem (see [3, 7]) the function  $\lambda(g, A)$  is well-defined and

$$\lambda(g, A)(x) \leq \text{Lip}(g, A).$$

Moreover, (see [3, Lemma 2.10.40]) for any  $x \in \mathbb{R}^n \setminus A$  there exists a unique  $y(x) \in \mathbb{R}^m$  such that

$$\lambda(g, A)(x) = \sup_{a \in A} \frac{\|g(a) - y(x)\|}{\|a - x\|}, \quad (4)$$

and  $y(x)$  belongs to the convex hull of the set

$$B = \{g(z) : z \in A \text{ and } \frac{\|g(z) - y(x)\|}{\|z - x\|} = \lambda(g, A)(x)\}.$$

Thus we can define

$$K(g, A)(x) := \begin{cases} g(x) & \text{if } x \in A; \\ \arg \min_{y \in \mathbb{R}^m} \sup_{a \in A} \frac{\|g(a) - y\|}{\|a - x\|} & \text{if } x \in \mathbb{R}^n \setminus A. \end{cases} \quad (5)$$

We say that  $K(g, A)(x)$  is the *Kirszbraun value* of  $g$  restricted on  $A$  at point  $x$ . The function  $K(g, A)(x)$  is the best extension at point  $x$  such that the Lipschitz constant is minimal. We produce a method to compute  $\lambda(g, A)(x)$  and  $K(g, A)(x)$  in section 4.

Let  $G = (V, E, \Omega)$  be a connected finite graph with vertices set  $V \subset \mathbb{R}^n$ , edges set  $E$  and a non-empty set  $\Omega \subset V$ .

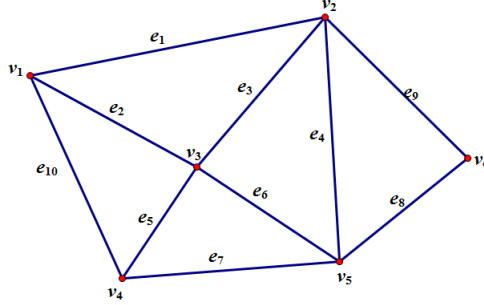


Figure 1: A simple picture of graph  $G$

For  $x \in V$ , we define

$$S(x) := \{y \in V : (x, y) \in E\} \quad (6)$$

to be the neighborhood of  $x$  on  $G$ .

**Example 1.** In Figure 1 we have  $V = \{v_1, \dots, v_6\}$ ,  $E = \{e_1, \dots, e_{10}\}$ ,  $S(v_3) = \{v_1, v_2, v_4, v_5\}$ .

Let  $f : \Omega \rightarrow \mathbb{R}^m$ . We consider the following functional equation with Dirichlet's condition:

$$u(x) = \begin{cases} K(u, S(x))(x) & \forall x \in V \setminus \Omega; \\ f(x) & \forall x \in \Omega. \end{cases} \quad (7)$$

We say that a function  $u$  satisfying (7) is a *Kirszbraun extension* of  $f$  on graph  $G$ . This extension is the optimal Lipschitz extension of  $f$  on graph  $G$  since for any  $x \in V \setminus \Omega$ , there is no way to decrease  $\text{Lip}(u, S(x))$  by changing the value of  $u$  at  $x$ .

In real valued case  $m = 1$ , the function  $K(u, S(x))(x)$  was considered by Oberman [9] and he used this function to obtain a convergent difference scheme for the AMLE. Le Gruyer [6] showed the explicit formula for  $K(u, S(x))(x)$  as follows

$$K(u, S(x))(x) = \inf_{z \in S(x)} \sup_{q \in S(x)} M(u, z, q)(x), \quad (8)$$

where

$$M(u, z, q)(x) := \frac{\|x - z\|u(q) + \|x - q\|u(z)}{\|x - z\| + \|x - q\|}.$$

Le Gruyer studied the solution of (7) on a network where  $K(u, S(x))(x)$  satisfying (8). This solution plays an important role in approximation arguments for AMLE in Le Gruyer [6].

The Kirszbraun extension  $u$  is a generalization of the solution in the previous works of Le Gruyer for vector valued cases ( $m \geq 2$ ). We prove that the tight function introduced by Sheffield and Smart (2012) [10] is a Kirszbraun extension. Therefore, we have the existence of a Kirszbraun extension, but in general Kirszbraun extension maybe not unique.

In the scalar case  $m = 1$ , Le Gruyer [6] defined a network on a metric space  $(X, d)$  as follows

**Definition 2.** A network on a metric space  $(X, d)$  is a couple  $(N, U)$  where  $N \subset X$  denotes a finite non-empty subset of  $\mathbb{R}^n$  and  $U$  a mapping  $x \in N \rightarrow U(x) \subset N$  which satisfies

- (P1) For any  $x \in N$ ,  $x \in U(x)$ .
- (P2) For any  $x, y \in N$ ,  $x \in U(y)$  iff  $y \in U(x)$ .
- (P3) For any  $x, y \in N$ , there exists  $x_1, \dots, x_{n-1} \in G$  such that  $x_1 = x$ ,  $x_n = y$  and  $x_i \in U(x_{i+1})$  for  $i = 1, \dots, n - 1$ .
- (P4) For any  $x \in N$ , any  $y \in N \setminus U(x)$  there exists  $z \in U(x)$  such that  $d(z, y) \leq d(x, y)$ .

In the above definition,  $U(x)$  is called the neighborhood of  $x$  on network  $(N, U)$ . Let  $g : A \subset X \rightarrow \mathbb{R}$ . In [6] Le Gruyer defined the Kirszbraun extension of  $g$  with respect to the network (see [6, page 30]) and he proved the existence and uniqueness of the Kirszbraun extension of  $g$  on the network. In particular, when  $X = \mathbb{R}^n$  equipped with the Euclidean norm, Le Gruyer obtained the approximation for AMLE by a sequence Kirszbraun extensions  $(u_n)$  of networks  $(N_n, U_n)$  ( $n \in \mathbb{N}$ ) having some good properties.

Similarly to Le Gruyer's result about the uniqueness of the Kirszbraun extension on a network, in this chapter we prove the uniqueness of the Kirszbraun extension  $u$  of  $f$  on graph  $G$  when  $m = 1$ . The graph is more general than the network in some sense since there are many graphs that do not satisfy (P4). Moreover, in the scalar case  $m = 1$ , we produce a simple algorithm which calculates efficiently the value of Kirszbraun extension  $u$  in polynomial time. This algorithm is similar to the algorithm produced by Lazarus et al. (1999) [8] when they calculate the Richman cost function. Assuming Jensen's hypotheses [5], since this algorithm computes exactly solution of (7) and by using the argument of Le Gruyer [6] (the approximation for AMLE by a sequence Kirszbraun extensions  $(u_n)$  of networks  $(N_n, U_n)$  ( $n \in \mathbb{N}$ )), we obtain a new method to approximate the viscosity solution of Equation  $\Delta_\infty u = 0$  under Dirichler's condition  $f$ .

In the above algorithm, the explicit formula of  $K(u, S(x))$  in (8) and the order structure of real number set play important role. The generalization of the algorithm to vector valued

case ( $m \geq 2$ ) is difficult since we do not know the explicit formula of  $K(u, S(x))$  when  $m \geq 2$  and  $\mathbb{R}^2$  does not have any useful order structure. Extending the results of the approximation of AMLE to vector valued cases ( $m \geq 2$ ) presents many difficulties which have limited the number of results in this direction, see [4] and the references therein.

## 2 The existence of Kirszbraun extension

In this section, we prove the existence of Kirszbraun extension satisfying Equation (7).

Let  $G = (V, E, \Omega)$  be a connected finite graph with vertices set  $V \subset \mathbb{R}^n$ , edges set  $E$  and a non-empty set  $\Omega \subset V$  and let  $f : \Omega \rightarrow \mathbb{R}^m$ .

We denote  $E(f)$  to be the set of all extensions of  $f$  on  $G$ .

Let  $v \in E(f)$ . The *local Lipschitz constant* of  $v$  at vertex  $x \in V \setminus \Omega$  is given by

$$Lv(x) := \sup_{y \in S(x)} \frac{\|v(y) - v(x)\|}{\|y - x\|},$$

where  $S(x)$  is neighborhood of  $x$  on  $G$ .

**Definition 3.** <sup>1</sup> If  $u, v \in E(f)$  satisfy

$$\max\{Lu(x) : Lu(x) > Lv(x), x \in V \setminus \Omega\} > \max\{Lv(x) : Lv(x) > Lu(x), x \in V \setminus \Omega\},$$

then we say that  $v$  is *tighter* than  $u$  on  $G$ . We say that  $u$  is a *tight extension* of  $f$  on  $G$  if there is no  $v$  tighter than  $u$ .

**Theorem 4.** [10, Theorem 1.2] *There exists a unique extension  $u$  that is tight of  $f$  on  $G$ . Moreover,  $u$  is tighter than every other extension  $v$  of  $f$ .*

**Proposition 5.** *Let  $u \in E(f)$ . Let  $x \in V \setminus \Omega$ , we define*

$$v(y) = \begin{cases} u(y), & \text{if } y \in V \setminus \{x\}, \\ K(u, S(x))(x), & \text{if } y = x. \end{cases}$$

*If  $K(u, S(x))(x) \neq u(x)$  then  $v$  is tighter than  $u$ .*

*Proof.*

**\*Step 1:** In this step we prove that for any  $y \in V \setminus \Omega$ , we obtain

$$Lv(y) \leq \max\{Lv(x), Lu(y)\}. \tag{9}$$

Indeed,

\*If  $y \notin S(x) \cup \{x\}$ . Since  $v(y) = u(y)$  and  $v(z) = u(z)$  for all  $z \in S(y)$ , we obtain

$$Lv(y) = Lu(y).$$

\*If  $y = x$ . Since  $v(x) \neq u(x)$  and  $v(x)$  is the Kirszbraun value of  $u$  restricted on  $S(x)$  at point  $x$ , we have

$$Lv(y) < Lu(y).$$

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<sup>1</sup>By convention, if  $C = \emptyset$  then  $\max_C = 0$ .

\*If  $y \in S(x)$  we have

$$\begin{aligned} Lv(y) &= \max_{z \in S(y)} \frac{\|v(z) - v(y)\|}{\|z - y\|} \\ &= \max_{z \in S(y) \setminus \{x\}} \left\{ \frac{\|v(x) - v(y)\|}{\|x - y\|}, \frac{\|u(z) - u(y)\|}{\|z - y\|} \right\} \\ &\leq \max\{Lv(x), Lu(y)\}. \end{aligned}$$

Therefore, for any  $y \in V \setminus \Omega$  we have

$$Lv(y) \leq \max\{Lv(x), Lu(y)\}.$$

**\*Step 2:** In this step we prove that  $v$  is tighter than  $u$ . It means that we need to show that

$$\max\{Lv(y) : Lv(y) > Lu(y), y \in V \setminus \Omega\} < \max\{Lu(y) : Lu(y) > Lv(y), y \in V \setminus \Omega\}$$

Indeed, if  $Lv(y) > Lu(y)$  then from (9) we have  $Lv(y) \leq Lv(x)$ . Thus

$$\max\{Lv(y) : Lv(y) > Lu(y), y \in V \setminus \Omega\} \leq Lv(x) \quad (10)$$

Since  $v(x) \neq u(x)$  and  $v(x)$  is the Kirszbraun value of  $u$  restricted on  $S(x)$  at point  $x$ , we have

$$Lv(x) < Lu(x). \quad (11)$$

From (10) and (11) we obtain

$$\begin{aligned} \max\{Lv(y) : Lv(y) > Lu(y), y \in V \setminus \Omega\} &\leq Lv(x) \\ &< Lu(x) \\ &\leq \max\{Lu(y) : Lu(y) > Lv(y), y \in V \setminus \Omega\}. \end{aligned}$$

□

We obtain the existence of a Kirszbraun extension satisfying Equation (7) as a consequence of the following theorem.

**Theorem 6.** *If  $u$  is a tight extension of  $f$ , then  $u$  is a Kirszbraun extension of  $f$ .*

*Proof.* Let  $u$  be a tight extension of  $f$ . Suppose, by contradiction, that there are some  $x \in V \setminus \Omega$  such that

$$K(u, S(x))(x) \neq u(x). \quad (12)$$

we define

$$v(y) = \begin{cases} u(y), & \text{if } y \in V \setminus \{x\}, \\ K(u, S(x)), & \text{if } y = x. \end{cases}$$

By applying Proposition 5 we have  $v$  tighter than  $u$ . This is impossible since  $u$  is tight of  $f$ . □

### 3 An algorithm to compute Kirszbraun extension when $m = 1$

In this section, let  $G = (V, E, \Omega)$  be a connected finite graph, with vertices set  $V \subset \mathbb{R}^n$ , edges set  $E$  and a non-empty set  $\Omega \subset V$ . Let  $f : \Omega \rightarrow \mathbb{R}$ .

We recall some properties of Kirszbraun function introduced in (5) which are useful in the proof of Theorem 8.

**Theorem 7.** Let  $S = \{x_1, \dots, x_n\} \subset \mathbb{R}^n$  and  $u : S \rightarrow \mathbb{R}$ . For each  $x \in \mathbb{R}^n \setminus S$ , we use the notation  $d_i = \|x_i - x\|$ ,  $i = 1, \dots, n$ .

(a) (see [9, Theorem 5]) We have

$$K(u, S)(x) = \frac{d_i u(x_j) + d_j u(x_i)}{d_i + d_j},$$

where  $i, j$  are the indexes which satisfy

$$\frac{|u(x_i) - u(x_j)|}{d_i + d_j} = \max_{k, l=1}^n \left\{ \frac{|u(x_k) - u(x_l)|}{d_k + d_l} \right\}.$$

(b) (see [3, Lemma 2.10.40]) Let

$$\lambda(u, S)(x) := \inf_{y \in \mathbb{R}^m} \sup_{a \in S} \frac{\|u(a) - y\|}{\|a - x\|} \text{ if } x \in \mathbb{R}^n \setminus S. \quad (13)$$

then the set

$$B = \left\{ u(z) : z \in S \text{ and } \frac{\|u(z) - K(u, S)(x)\|}{\|z - x\|} = \lambda(u, S)(x) \right\},$$

is not empty, and  $K(u, S)(x)$  belongs to the convex hull of  $B$ .

**Theorem 8.** There is a unique Kirszbraun extension  $u$  of  $f$  on the graph  $G$ . Moreover, the Kirszbraun extension  $u$  of  $f$  can be calculated in polynomial time.

Before proving Theorem 8, we need the following definition

**Definition 9.** Let  $G' = (V', E', \Omega)$  be a subgraph of  $G$ , i.e.  $\Omega \subset V' \subset V$  and  $E' \subset E$ . Let  $u'$  be a Kirszbraun extension of  $f$  on  $G'$ , a *connecting path on  $G'$  with respect to  $u'$*  is a sequence

$$v_0, e_1, v_1, \dots, e_n, v_n \quad (n \geq 1)$$

of distinct vertices and edges in  $G$  such that

- \* each  $e_i$  is an edge joining  $v_{i-1}$  and  $v_i$ ,
- \*  $v_0$  and  $v_n$  are in  $V'$ ,
- \* for  $1 \leq i < n$ ,  $v_i$  is in  $V \setminus V'$ , and
- \* for  $1 \leq i \leq n$ ,  $e_i$  is in  $E \setminus E'$

We define

$$c := \frac{|u'(v_n) - u'(v_0)|}{\sum_{i=1}^n \|v_i - v_{i-1}\|}.$$

We say that  $c$  is the *slope* of the connecting path  $v_0, e_1, v_1, \dots, e_n, v_n$ .

**Proof of Theorem 8.** We construct an increasing sequence of subgraph  $G_n = (V_n, E_n, \Omega)$  of  $G$  and  $u_n$  which is a Kirszbraun extension of  $f$  on  $G_n$ . We finish the algorithm with a Kirszbraun extension  $u$  on  $G$ .

**Step 1: Construct an increasing sequence of subgraph**

We begin with the trivial subgraph  $G_1 = (V_1, E_1, \Omega)$  where  $V_1 = \Omega$ ,  $E_1 = \emptyset$  and let  $u_1 = f$  on  $\Omega$ . It is clear that  $u_1$  is a Kirszbraun extension of  $f$  on  $G_1$ . The algorithm then proceeds in stages.

Suppose that after  $n$  stages we have an increasing sequence of subgraph  $G_l = (V_l, E_l, \Omega)$  of  $G$  and  $u_l$  is a Kirszbraun extension of  $f$  on  $G_l$  for  $l = 1, \dots, n$ .

If there are no connecting paths on  $G_n$  with respect to  $u_n$ , we go to step 2.

If there are some connecting paths on  $G_n$  with respect to  $u_n$ . We construct  $G_{n+1}$  subgraph of  $G$  and  $u_{n+1}$  Kirszbraun extension of  $f$  on  $G_{n+1}$  as follows:

Find a connecting path  $v_0, e_1, v_1, \dots, e_k, v_k$  ( $k \geq 1$ ) on  $G_n$  with respect to  $u_n$  with largest possible slope  $c_n$ .

Without loss of generality, we label the vertices of the path so that  $u_n(v_k) \geq u_n(v_0)$ . We define

$$\begin{aligned} u_{n+1}(x) &:= \begin{cases} u_n(x), & \forall x \in G_n \\ u_n(v_0) + c_n \sum_{j=1}^i \|v_j - v_{j-1}\|, & \text{if } x = v_i \text{ for } i = 1, \dots, k-1. \end{cases} \quad (14) \\ V_{n+1} &:= V_n \cup \{v_1, \dots, v_{k-1}\} \\ E_{n+1} &:= E_n \cup \{e_1, \dots, e_k\} \end{aligned}$$

We will show that  $u_{n+1}$  is a Kirszbraun extension of  $f$  on graph  $G_{n+1} = (V_{n+1}, E_{n+1}, \Omega)$ .

For  $x \in V_{n+1}$ , let

$$S_i(x) := \{y \in V_i : (x, y) \in E_i\} \quad \text{for } i \in \{1, \dots, n+1\}.$$

be the neighborhood of  $x$  with respect to  $G_i$ .

**Case 1:**  $x \in V_n \setminus \{v_0, v_k\}$ .

We have  $S_{n+1}(x) = S_n(x)$ ,  $u_{n+1}(z) = u_n(z)$  for all  $z \in S_{n+1}(x) \cup \{x\}$  and  $u_n(x) = K(u_n, S_n(x))(x)$  since  $u_n$  is Kirszbraun of  $G_n$ . Thus

$$u_{n+1}(x) = K(u_{n+1}, S_{n+1}(x))(x), \quad \text{for } x \in V_n \setminus \{v_0, v_k\}.$$

**Case 2:**  $x \in \{v_1, \dots, v_{k-1}\}$ .

Noting that  $S_{n+1}(v_i) = \{v_{i-1}, v_{i+1}\}$  for all  $i = 1, \dots, k-1$ . Moreover, from (14), we have

$$\frac{u_{n+1}(v_i) - u_{n+1}(v_{i-1})}{\|v_i - v_{i-1}\|} = c_n, \quad \forall i : 1 \leq i \leq k.$$

Hence

$$u_{n+1}(x) = K(u_{n+1}, S_{n+1}(x))(x) \quad \forall x \in \{v_1, \dots, v_{k-1}\}.$$

**Case 3:**  $x \in \{v_0, v_k\}$ .

We need to prove that

$$u_{n+1}(v_0) = K(u_{n+1}, S_{n+1}(v_0))(v_0). \quad (15)$$

(Proving  $u_{n+1}(v_k) = K(u_{n+1}, S_{n+1}(v_k))(v_k)$  is similar.)

To see (15), we must show that

$$\sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} \leq \inf_{y \in \mathbb{R}^m} \sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - y|}{\|x - v_0\|} \quad (16)$$

Noting that  $u_{n+1}(x) = u_n(x)$  for all  $x \in S_n(v_0) \cup \{v_0\}$ ,  $S_{n+1}(v_0) = S_n(v_0) \cup \{v_1\}$  and  $c_n = \frac{|u_{n+1}(v_1) - u_{n+1}(v_0)|}{\|v_1 - v_0\|}$ . Moreover, since  $u_n$  is a Kirszbraun extension of  $f$  on  $G_n$ , we have

$$\sup_{x \in S_n(v_0)} \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|} \leq \inf_{y \in \mathbb{R}^m} \sup_{x \in S_n(v_0)} \frac{|u_n(x) - y|}{\|x - v_0\|}.$$

Thus

$$\begin{aligned} \sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} &= \sup_{x \in S_n(v_0) \cup \{v_1\}} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} \\ &= \max_{x \in S_n(v_0)} \left\{ \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|}, \frac{|u_{n+1}(v_1) - u_{n+1}(v_0)|}{\|v_1 - v_0\|} \right\} \\ &= \max \left\{ \max_{x \in S_n(v_0)} \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|}, c_n \right\}, \end{aligned}$$

and

$$\begin{aligned} \max_{x \in S_n(v_0)} \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|} &\leq \inf_{y \in \mathbb{R}^m} \sup_{x \in S_n(v_0)} \frac{|u_n(x) - y|}{\|x - v_0\|} \\ &\leq \inf_{y \in \mathbb{R}^m} \sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - y|}{\|x - v_0\|}. \end{aligned}$$

Therefore, to obtain Equation (16), we need to prove that

$$c_n \leq \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|}, \quad (17)$$

for some  $x \in S_n(v_0)$ .

Let  $\mathcal{F}$  be the set of slope of connecting paths occurring in the algorithm. Remark that each edges and each vertices entered in our algorithm relate with a slope in  $\mathcal{F}$ . So that, for any  $y \in V_n$ , there exist some  $x \in S_n(x)$  and  $c \in \mathcal{F}$  such that

$$c = \frac{|u_n(x) - u_n(y)|}{\|x - y\|}. \quad (18)$$

From above remark, to see (17), we need to show that the sequence of slope of connecting paths occurring in the algorithm is non-increasing. We show this in our present notation. Suppose that

$$w_0, f_1, w_1, \dots, f_m, w_m \quad (m \geq 1)$$



is a connecting path on  $G_{n+1}$  with respect to  $u_{n+1}$  with slope  $c_{n+1}$ . We need to prove that  $c_n \geq c_{n+1}$ . We assume without loss of generality that  $u_{n+1}(w_0) \leq u_{n+1}(w_m)$ .

- If  $w_0$  and  $w_m$  are both in  $V_n$  then the connecting path on  $G_{n+1}$  with respect to  $u_{n+1}$  is actually the connecting path on  $G_n$  with respect to  $u_n$ . Therefore, since  $c_n$  is the largest slope of connecting paths on  $G_n$  with respect to  $u_n$ , we have  $c_n \geq c_{n+1}$ .

- If  $w_0 = v_i$  and  $w_m = v_j$  for some  $0 \leq i < j \leq k$ . We consider the path through the vertices

$$v_0, \dots, v_{i-1}, w_0, \dots, w_m, v_{j+1}, \dots, v_k.$$

The slope of above path is

$$c = \frac{|u_n(v_k) - u_n(v_0)|}{\sum_{l=1}^i \|v_l - v_{l-1}\| + \sum_{l=1}^m \|w_l - w_{l-1}\| + \sum_{l=j+1}^k \|v_l - v_{l-1}\|}.$$

Since  $c_n$  is the largest slope of connecting paths on  $G_n$  with respect to  $u_n$ , we have  $c_n \geq c$ . Moreover,

$$c_n = \frac{|u_n(v_k) - u_n(v_0)|}{\sum_{l=1}^k \|v_l - v_{l-1}\|},$$

thus we obtain

$$\sum_{l=1}^m \|w_l - w_{l-1}\| \geq \sum_{l=i+1}^j \|v_l - v_{l-1}\|.$$

Hence

$$c_{n+1} = \frac{|u_{n+1}(w_m) - u_{n+1}(w_0)|}{\sum_{k=1}^m \|w_k - w_{k-1}\|} = \frac{|u_{n+1}(v_j) - u_{n+1}(v_i)|}{\sum_{k=1}^m \|w_k - w_{k-1}\|} \leq \frac{|u_{n+1}(v_j) - u_{n+1}(v_i)|}{\sum_{l=i+1}^j \|v_l - v_{l-1}\|} = c_n.$$

## Step 2: Completing the algorithm

If there are no connecting paths on  $G_n = (V_n, E_n, \Omega)$  with respect to  $u_n$ . Then each unlabeled vertex  $v$  is connected via edges not in  $E_n$  to exactly one vertex  $w$  of  $V_n$ . We extend  $u_n$  to the point  $w$  by putting  $u_n(w) := u_n(v)$ . This completes the algorithm, and we obtains a Kirszbraun extension of  $f$ .

Since each stage adds at least one edge, and each stage can be accomplished by one shortest-path search for each pair of labeled vertices, this algorithm is calculated in polynomial time.

## Uniqueness

Let  $u$  be the Kirszbraun extension of  $f$  defined by the algorithm above and  $h$  be another Kirszbraun extension of  $f$ . Let  $v$  be the first vertex added by algorithm such that  $u(v) \neq h(v)$ .

• If  $v$  is added to a subgraph  $G' = (V', E', \Omega)$  as part of a connecting path through the vertices

$$v_0, \dots, v_k, \dots, v_n$$

with slope  $c$  and  $v = v_k$ .

We can assume without loss of generality that  $u(v_0) \leq u(v_n)$ . Let

$$\mathcal{L} = \{v_i : 0 \leq i \leq n, h(v_i) \geq u(v_i), h(v_i) - h(v_{i-1}) > u(v_i) - u(v_{i-1})\}$$

We prove that  $\mathcal{L} \neq \emptyset$ . Indeed, by contradiction, suppose that  $\mathcal{L} = \emptyset$ . Since  $u(v_0) = h(v_0)$  and  $\mathcal{L} = \emptyset$  we must have

$$h(v_1) \leq u(v_1).$$

If  $h(v_2) > u(v_2)$  then

$$h(v_2) - h(v_1) > u(v_2) - u(v_1).$$

Hence  $v_2 \in \mathcal{L}$ . This contradicts with  $\mathcal{L} = \emptyset$ . Thus we must have

$$h(v_2) \leq u(v_2).$$

By induction, we have

$$h(v_i) \leq u(v_i) \quad \forall i : 0 \leq i \leq k. \quad (19)$$

Since  $v = v_k$ ,  $h(v) \neq u(v)$  and (19), we have  $h(v_k) < u(v_k)$ . Thus if  $h(v_{k+1}) \geq u(v_{k+1})$  then

$$h(v_{k+1}) - h(v_k) > u(v_{k+1}) - u(v_k).$$

Hence  $v_{k+1} \in \mathcal{L}$ . This contradicts with  $\mathcal{L} = \emptyset$ . Thus we must have

$$h(v_{k+1}) < u(v_{k+1}).$$

By induction, we have

$$h(v_i) < u(v_i), \quad \forall k \leq i \leq n.$$

But we know that  $h(v_n) = u(v_n)$ , thus we have a contradiction. Therefore  $\mathcal{L} \neq \emptyset$ .

Let  $v_l \in \mathcal{L}$ . We have

$$\begin{cases} h(v_l) \geq u(v_l); \\ h(v_l) - h(v_{l-1}) > u(v_l) - u(v_{l-1}). \end{cases} \quad (20)$$

Hence

$$\Delta := \frac{h(v_l) - h(v_{l-1})}{\|v_l - v_{l-1}\|} > \frac{u(v_l) - u(v_{l-1})}{\|v_l - v_{l-1}\|} = c \geq 0. \quad (21)$$

Set

$$S(x) := \{y \in V, (x, y) \in E\} \quad , \text{ for } x \in V.$$

Since  $K(h, S(v_l))(v_l) = h(v_l)$ , by applying Theorem 7, there exists  $z_1 \in S(v_l)$  such that

$$\frac{h(z_1) - h(v_l)}{\|z_1 - v_l\|} = \max\left\{\frac{h(y) - h(v_l)}{\|y - v_l\|} : y \in S(v_l)\right\}.$$

Thus

$$\frac{h(z_1) - h(v_l)}{\|z_1 - v_l\|} \geq \frac{h(v_l) - h(v_{l-1})}{\|v_l - v_{l-1}\|} = \Delta.$$

We extend *path of greatest*  $z_1, z_2, \dots$  such that  $z_{j+1} \in S(z_j)$  and

$$\frac{h(z_{j+1}) - h(z_j)}{\|z_{j+1} - z_j\|} = \max\left\{\frac{h(y) - h(z_j)}{\|y - z_j\|} : y \in S(z_j)\right\} \geq \Delta.$$

This path must terminate with a  $z_m \in V'$ .

Since  $\Delta > 0$ , we have

$$h(z_m) > \dots > h(v_l) \geq u(v_l) \geq u(v_0).$$

Thus  $z_m \neq v_0$ .

Finally, consider the path through the vertices

$$v_0, v_1, \dots, v_l, z_1, \dots, z_m.$$

Set  $z_0 := v_l$ . The above path is the connecting path on  $G'$  with respect to  $u$ .

Moreover,  $c$  is the largest slope of connecting paths on  $V'$  with respect to  $u$ , and

$$u(v_0) = h(z_0), \quad u(z_m) = h(z_m), \quad h(z_0) = h(v_l) \geq u(v_l),$$

$$\frac{h(z_{i+1}) - h(z_i)}{\|z_{i+1} - z_i\|} \geq \Delta, \quad \frac{u(v_l) - u(v_0)}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} = c, \quad \text{and } \Delta > c.$$

Thus we have

$$\begin{aligned} c &\geq \frac{u(z_m) - u(v_0)}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum_{i=0}^{m-1} \|z_{i+1} - z_i\|} \geq \frac{h(z_m) - h(z_0) + u(v_l) - u(v_0)}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum_{i=0}^{m-1} \|z_{i+1} - z_i\|} \\ &= \sum_{i=0}^{m-1} \frac{h(z_{i+1}) - h(z_i)}{\|z_{i+1} - z_i\|} \cdot \frac{\|z_{i+1} - z_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum_{i=0}^{m-1} \|z_{i+1} - z_i\|} \\ &\quad + \frac{u(v_l) - u(v_0)}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} \cdot \frac{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum_{i=0}^{m-1} \|z_{i+1} - z_i\|} \\ &\geq \Delta \frac{\sum_{i=0}^{m-1} \|z_{i+1} - z_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum_{i=0}^{m-1} \|z_{i+1} - z_i\|} + c \cdot \frac{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\| + \sum_{i=0}^{m-1} \|z_{i+1} - z_i\|} \\ &> c \end{aligned}$$

The last inequality is obtained by  $\Delta > c$ . Thus we have a contradiction.

- If  $v$  is added during the final step of the algorithm. We call  $G' = (V', E', \Omega)$  to be the subgraph of  $G = (V, E, \Omega)$  when we finish step 1 in the algorithm. Thus there are no connecting paths on  $G'$  with respect to  $u$ . Therefore,  $v$  is connected via edges not in  $E'$  to exactly one vertex  $w$  of  $V'$ .

We can find the largest connected subgraph  $G'' = (V'', E'', \Omega)$  satisfying

$$v, w \in V'', V'' \cap V' = \{w\}, \quad \text{and} \quad E'' \cap E' = \emptyset.$$

From the definition of  $u$ , we have

$$h(w) = u(w) = u(x), \quad \forall x \in V''.$$

Since  $u(v) \neq h(v)$  and  $h(w) = u(w) = u(v)$ , we have  $h(w) \neq h(v)$ . Therefore, we must have  $\sup_{z \in V''} h(z) \neq h(w)$  or  $\inf_{z \in V''} h(z) \neq h(w)$ .

Suppose  $\sup_{z \in V''} h(z) \neq h(w)$  (we prove similar for the case  $\inf_{z \in V''} h(z) \neq h(w)$ ). Let  $v_0 \in V''$  such that

$$h(v_0) = \sup_{z \in V''} h(z) \neq h(w).$$

Set

$$S''(x) := \{y \in V'' : (x, y) \in E''\}, \quad \text{for } x \in V'' \setminus \{w\},$$

and

$$S(x) := \{y \in V : (x, y) \in E\}, \quad \text{for } x \in V \setminus \Omega.$$

Noting that

$$S(x) = S''(x), \quad \forall x \in V'' \setminus \{w\}. \tag{22}$$

Since  $G''$  is a connected graph, there exists a path through the vertices

$$v_0, v_1, \dots, v_n, w$$

such that  $v_i \in S''(v_{i-1}), \forall i \in \{1, \dots, n\}$  and  $w \in S''(v_n)$ .

On the other hand, from (22) and since  $h$  is Kirszbraun extension, we have

$$h(v_0) = \sup_{z \in V''} h(z) \geq \sup_{z \in S''(v_0)} h(z) = \sup_{z \in S(v_0)} h(z).$$

Thus applying Theorem 7 we have

$$h(v_0) = h(s), \quad \forall s \in S(v_0).$$

In particular, we have  $h(v_0) = h(v_1)$ . By induction, we obtain

$$h(v_0) = h(v_1) = \dots = h(v_n) = h(w).$$

This contradicts with  $h(w) \neq h(v_0)$ . □

**Remark 10.** Assuming Jensen's hypotheses [5], since this algorithm computes exactly solution of (7) and by using the argument of Le Gruyer [6] (the approximation for AMLE by a sequence Kirszbraum extensions  $(u_n)$  of networks  $(N_n, U_n)$  ( $n \in \mathbb{N}$ )), we obtain a new method to approximate the viscosity solution of Equation  $\Delta_\infty u = 0$  under Dirichler's condition  $f$ .

**Definition 11.** For any  $x, y \in V$ . There exists a chain  $x_1, \dots, x_n \in V$  such that  $x_1 = x, x_n = y$  and  $x_i \in S(x_{i+1})$  for  $i = 1, \dots, n-1$ . To any chain we associate its length  $\sum_{i=1}^{n-1} \|x_i - x_{i+1}\|$ . We define the geodesis metric  $d_g$  of Graph  $G$  by letting  $d_g(x, y)$  be the infimum of the length of chains connecting  $x$  and  $y$ .

By using induction respect to increasing sequence of subgraph in the algorithm, we obtain the following theorem.

**Theorem 12.** Let  $u$  be the Kirszbraum extension of  $f$ . We have

$$\sup_{x, y \in V} \frac{\|u(x) - u(y)\|}{d_g(x, y)} \leq \sup_{x, y \in \Omega} \frac{\|f(x) - f(y)\|}{d_g(x, y)},$$

and

$$\inf_{z \in \Omega} f(z) \leq u(x) \leq \sup_{z \in \Omega} f(z), \quad \forall x \in V.$$

## 4 Method to find $K(f, S)(x)$ in general case for any $m \geq 1$

We fix  $S = \{p_1, \dots, p_N\} \subset \mathbb{R}^n$  and  $f : S \rightarrow \mathbb{R}^m$  to be a Lipschitz function. Let  $x \in \mathbb{R}^n \setminus S$ . We denote

$$\lambda(f, S)(x) := \inf_{y \in \mathbb{R}^m} \sup_{i \in \{1, \dots, N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|}.$$

By applying Kirszbraum's theorem (see [3, 7]) we have  $\lambda \leq \text{Lip}(f, S)$ .

In this section, we show a method to compute  $\lambda(f, S)(x)$  and  $K(f, S)(x)$  given by (5).

We recall some results that will be useful in this section.

**Lemma 13.** ([3, Lemma 2.10.40]) There exists a unique  $y(x) \in \mathbb{R}^m$  such that

$$\lambda(f, S)(x) = \sup_{a \in S} \frac{\|f(a) - y(x)\|}{\|a - x\|}, \quad (23)$$

and  $y(x)$  belongs to the convex hull of the set

$$B = \{f(z) : z \in S \text{ and } \frac{\|f(z) - y(x)\|}{\|z - x\|} = \lambda(f, S)(x)\}.$$

Moreover, from the definition of  $K(f, S)(x)$ , we have  $K(f, S)(x) = y(x)$ .

To compute the value of  $K(f, S)(x)$  we need some properties of Cayley-Menger determinant. We recall some definitions and basic results.

Let  $x_1, \dots, x_k \in \mathbb{R}^n$ . We define the Cayley-Menger determinant of  $(x_i)_{i=1, \dots, k}$  as

$$\Gamma(x_1, \dots, x_k) := \det \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & \|x_1 - x_2\|^2 & \dots & \|x_1 - x_k\|^2 \\ 1 & \|x_2 - x_1\|^2 & 0 & \dots & \|x_2 - x_k\|^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \|x_k - x_1\|^2 & \|x_k - x_2\|^2 & \dots & 0 \end{pmatrix}.$$

**Definition 14.** A  $k$ -simplex is a  $k$ -dimensional polytope which is the convex hull of its  $k+1$  vertices. More formally, suppose the  $k+1$  points  $u_0, \dots, u_k \in \mathbb{R}^n$  are affinely independent, which means  $u_1 - u_0, \dots, u_k - u_0$  are linearly independent. Then the  $k$ -simplex determined by them is the set of points

$$C = \{t_0 u_0 + \dots + t_k u_k : t_i \geq 0, 0 \leq i \leq k, \sum_{i=0}^k t_i = 1\}.$$

**Example 15.** A 2-simplex is a triangle, a 3-simplex is a tetrahedron.

The  $k$ -simplex and the Cayley-Menger determinant have beautiful relations by following theorem:

**Theorem 16.** [2, Lemma 9.7.3.4] Let  $(x_i)_{i=1, \dots, k+2} \in \mathbb{R}^n$  be arbitrary points in  $k$ -dimensional Euclidean affine space  $X$ . Then  $\Gamma(x_1, \dots, x_{k+2}) = 0$ . A necessary and sufficient condition for  $(x_i)_{i=1, \dots, k+1}$  to be a  $k$ -simplex of  $X$  is that  $\Gamma(x_1, \dots, x_{k+1}) \neq 0$ .

**Lemma 17.** Let the point  $u$  lie in the convex hull of the points  $q_0, q_1, \dots, q_s$  of  $\mathbb{R}^m$ . If  $u'$  distinct from  $u$ , then for some  $i$ :

$$\|u - q_i\| \leq \|u' - q_i\|.$$

*Proof.* Choose  $H$  to be the  $(m-1)$ -dimension (or hyperplane) through  $u$  which is perpendicular to the segment  $[u, u']$ . Then for at least one value for  $i$ ,  $q_i$  must lie in the halfspace of  $H$  which does not contain  $u'$ . Thus we have

$$\|u - q_i\| \leq \|u' - q_i\|.$$

□

**Proposition 18.** Suppose there exist  $J \subset \{1, 2, \dots, N\}$ ,  $f_0$  inside convex hull of  $\{f(p_j)\}_{j \in J}$  and  $\lambda_0 > 0$  such that

$$\|f_0 - f(p_j)\| = \lambda_0 \|x - p_j\|, \quad \forall j \in J$$

and

$$\|f_0 - f(p_i)\| \leq \lambda_0 \|x - p_i\|, \quad \forall i \in \{1, \dots, N\},$$

then  $\lambda_0 = \lambda(f, S)(x)$  and  $f_0 = K(f, S)(x)$ .

*Proof.* We have

$$\lambda_0 = \sup_{i \in \{1, \dots, N\}} \frac{\|f_0 - f(p_i)\|}{\|x - p_i\|} \geq \inf_{y \in \mathbb{R}^m} \sup_{i \in \{1, \dots, N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|} = \lambda(f, S)(x).$$

On the other hand, for any  $y \in \mathbb{R}^m$ , by applying Lemma 17 there exists  $i \in J$  such that

$$\|y - f(p_i)\| \geq \|f_0 - f(p_i)\| = \lambda_0 \|x - p_i\|.$$

Hence

$$\sup_{i \in \{1, \dots, N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|} \geq \lambda_0. \quad (24)$$

Since Inequality (24) is true for any  $y \in \mathbb{R}^m$ , we have  $\lambda(f, S)(x) \geq \lambda_0$ . Thus

$$\lambda(f, S)(x) = \lambda_0.$$

Therefore, we have

$$\lambda(f, S)(x) = \sup_{i \in \{1, \dots, N\}} \frac{\|f_0 - f(p_i)\|}{\|x - p_i\|}.$$

From Lemma 13 we have  $f_0 = K(f, S)(x)$ . □

### **A method to compute $K(f, S)(x)$**

Recall that  $f : S \rightarrow \mathbb{R}^m$ . By applying Lemma 13, we have

$$\|f(a) - K(f, S)(x)\| \leq \lambda(f, S)(x) \|a - x\|, \quad \forall a \in S.$$

Moreover,

$$B = \left\{ f(a) : a \in S \text{ and } \frac{\|f(a) - K(f, S)(x)\|}{\|a - x\|} = \lambda(f, S)(x) \right\},$$

is not empty, and  $K(f, S)(x)$  belongs to the convex hull of  $B$ .

Therefore, there exist  $\{f(p_{i_k})\}_{k=1, \dots, l+1} \subset f(S)$  such that

(I)  $l \leq m$ , where  $m$  is dimension of  $\mathbb{R}^m$ ;

(II)  $\{f(p_{i_k})\}_{k=1, \dots, l+1}$  is a  $l$ -simplex. From Theorem 16,  $\{f(p_{i_k})\}_{k=1, \dots, l+1}$  is a  $l$ -simplex to be equivalent to

$$\Gamma(K(f, S)(x), f(p_{i_1}), \dots, f(p_{i_{l+1}})) \neq 0; \quad (25)$$

(III)  $K(f, S)(x)$  belongs convex hull of  $\{f(p_{i_k})\}_{k=1, \dots, l+1}$ ;

(IV)

$$\|K(f, S)(x) - f(p_{i_k})\| = \lambda(f, S)(x) \|x - p_{i_k}\|, \quad \forall k = 1, \dots, l+1. \quad (26)$$

(V)

$$\|f(a) - K(f, S)(x)\| \leq \lambda(f, S)(x) \|a - x\|, \quad \forall a \in S.$$

From the above observations, we obtain

**Theorem 19.** *There exist  $\{f(p_{i_k})\}_{k=1,\dots,l+1} \subset f(S)$  ( $1 \leq l \leq m$ , where  $m$  is dimension of  $\mathbb{R}^m$ ),  $f_{i_1 i_2 \dots i_{l+1}} \in \mathbb{R}^m$  and  $\lambda_{i_1 i_2 \dots i_{l+1}} \in \mathbb{R}$  satisfying some following properties*

- (a)  $f_{i_1 i_2 \dots i_{l+1}}$  inside convex hull of  $\{f(p_{i_k})\}_{k=1,\dots,l+1}$ .
  - (b)  $\Gamma(f(p_{i_1}), f(p_{i_2}), \dots, f(p_{i_{l+1}})) \neq 0$ .
  - (c)  $\|f_{i_1 i_2 \dots i_{l+1}} - f(p_k)\| = \lambda_{i_1 i_2 \dots i_{l+1}} \|x - p_k\|$ ,  $\forall k \in \{i_1, i_2, \dots, i_{l+1}\}$ .
  - (d)  $\|f_{i_1 i_2 \dots i_{l+1}} - f(p_k)\| \leq \lambda_{i_1 i_2 \dots i_{l+1}} \|x - p_k\|$ ,  $\forall k \in \{1, \dots, N\}$ .
- Moreover, from Proposition 18 we have  $f_{i_1 i_2 \dots i_{l+1}} = K(f, S)(x)$  and  $\lambda_{i_1 i_2 \dots i_{l+1}} = \lambda(f, S)(x)$ .

Therefore, to compute the value of  $\lambda(u, S)(x)$  and  $K(u, S)(x)$ , we need to find  $\{f(p_{i_k})\}_{k=1,\dots,l+1} \subset f(S)$ ,  $f_{i_1 i_2 \dots i_{l+1}} \in \mathbb{R}^m$  and  $\lambda_{i_1 i_2 \dots i_{l+1}} \in \mathbb{R}$  satisfying the conditions (a),(b),(c),(d). We can do that step by step as follows

**\*Step 1:** For all  $i, j \in \{1, \dots, N\}$ , ( $i \neq j$ ). Let

$$\begin{aligned} f_{ij} &:= \frac{\|x - p_j\|}{\|x - p_i\| + \|x - p_j\|} f(p_i) + \frac{\|x - p_i\|}{\|x - p_i\| + \|x - p_j\|} f(p_j); \\ \lambda_{ij} &:= \frac{\|f(p_i) - f(p_j)\|}{\|x - p_i\| + \|x - p_j\|}. \end{aligned}$$

We have  $f_{ij}$  inside convex hull of  $\{f(p_i), f(p_j)\}$  and

$$\|f_{ij} - f(p_k)\| = \lambda_{ij} \|x - p_k\|, \quad \text{for } k \in \{i, j\}.$$

Test the following condition

$$\|f_{ij} - f(p_k)\| \leq \lambda_{ij} \|x - p_k\|, \quad \forall k \in \{1, \dots, N\} \quad (27)$$

If  $(i, j)$  satisfies the above condition, then from Proposition 18 we have  $f_{ij} = K(f, S)(x)$  and  $\lambda_{ij} = \lambda(f, S)(x)$ . We finish. If there is no  $(i, j) \in \{1, \dots, N\}$ , ( $i \neq j$ ) that satisfies the above condition, then we go to step 2.

**\*Step 2:** For all  $(i, j, k) \in \{1, \dots, N\} \times \{1, \dots, N\} \times \{1, \dots, N\}$ . Test the following condition

$$\Gamma(f(p_i), f(p_j), f(p_k)) \neq 0. \quad (28)$$

Let  $A$  is the set of all  $(i, j, k)$  that satisfies (28). We consider a  $(i, j, k) \in A$ . Thus from Theorem 16 we have

- $\{f(p_i), f(p_j), f(p_k)\}$  is 2-simplex.
- For any  $f_{ijk}$  inside convex hull of  $\{f(p_i), f(p_j), f(p_k)\}$  we have

$$\Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0.$$

We consider the following equations

$$\begin{cases} \Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0; \\ \|f_{ijk} - f(p_l)\| = \lambda_{ijk} \|x - p_l\|, \quad \forall l \in \{i, j, k\}; \end{cases}$$

We replace  $\|f_{ijk} - f(p_l)\|$  by  $\lambda_{ijk} \|x - p_l\|$  into the equation

$$\Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0.$$



We obtain that

$$\begin{aligned}
0 &= \Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) \\
&= \det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \|f_{ijk} - f_i\|^2 & \|f_{ijk} - f_j\|^2 & \|f_{ijk} - f_k\|^2 \\ 1 & \|f_i - f_{ijk}\|^2 & 0 & \|f_i - f_j\|^2 & \|f_i - f_k\|^2 \\ 1 & \|f_j - f_{ijk}\|^2 & \|f_j - f_i\|^2 & 0 & \|f_j - f_k\|^2 \\ 1 & \|f_k - f_{ijk}\|^2 & \|f_k - f_i\|^2 & \|f_k - f_j\|^2 & 0 \end{pmatrix} \\
&= \det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \lambda_{ijk}^2 \|x - p_i\|^2 & \lambda_{ijk}^2 \|x - p_j\|^2 & \lambda_{ijk}^2 \|x - p_k\|^2 \\ 1 & \lambda_{ijk}^2 \|x - p_i\|^2 & 0 & \|f_i - f_j\|^2 & \|f_i - f_k\|^2 \\ 1 & \lambda_{ijk}^2 \|x - p_j\|^2 & \|f_j - f_i\|^2 & 0 & \|f_j - f_k\|^2 \\ 1 & \lambda_{ijk}^2 \|x - p_k\|^2 & \|f_k - f_i\|^2 & \|f_k - f_j\|^2 & 0 \end{pmatrix} \\
&= a(x)\lambda^4 + b(x)\lambda^2 + c(x),
\end{aligned}$$

where  $a(x), b(x), c(x)$  are function only depending on  $x$  and initial data  $x_l, f(p_l)$  for  $l \in \{i, j, k\}$ .

By solving the equation

$$a(x)\lambda_{ijk}^4 + b(x)\lambda_{ijk}^2 + c(x) = 0, \quad (29)$$

we obtain that  $\lambda_{ijk}$  is a positive real root of the above polynomial. It maybe that Equation (29) have no any positive real root. In this case, we consider another  $(i', j', k') \in A$  until Equation (29) with respect to  $(i', j', k')$  have a positive real root. We call  $L$  is the set of all positive real root of equation (29).

Let  $\lambda_{ijk} \in L$ . We find  $f_{ijk}$  by solving the equations

$$\|f_{ijk} - f(p_l)\| = \lambda_{ijk}\|x - p_l\|, \quad \forall l \in \{i, j, k\}. \quad (30)$$

After that, we test the condition  $f_{ijk}$  in convex hull of  $\{f(p_l)\}_{l \in \{i, j, k\}}$ , and test the following condition

$$\|f_{ijk} - f(p_l)\| \leq \lambda_{ijk}\|x - p_l\|, \quad \forall l \in \{1, \dots, N\}. \quad (31)$$

If we has a  $\lambda_{ijk} \in L$  such that  $f_{ijk}$  in convex hull of  $\{f(p_l)\}_{l \in \{i, j, k\}}$  satisfying Equations (30) and Inequalities (31) then from Proposition 18 we have  $f_{ijk} = K(f, S)(x)$  and  $\lambda_{ijk} = \lambda(f, S)(x)$ . We finish. If there is no  $(i, j, k) \in A$  that satisfies the above conditions, then we go to step 3.

**\*Step 3:** By the similar way as step 2 for  $(i, j, k, l), (i, j, k, l, h), \dots$  until we can find a  $(i_1, \dots, i_k) \subset \{1, \dots, N\}$  such that  $f_{i_1, i_2, \dots, i_k}$  and  $\lambda_{i_1, i_2, \dots, i_k}$  satisfying some following properties

- (a)  $f_{i_1, i_2, \dots, i_k}$  inside convex hull of  $\{f(p_{i_n})\}_{n=1, \dots, k}$
- (b)  $\Gamma(f(p_{i_1}), f(p_{i_2}), \dots, f(p_{i_k})) \neq 0$ .
- (c)  $\|f_{i_1, i_2, \dots, i_k} - f(p_l)\| = \lambda_{i_1, i_2, \dots, i_k}\|x - p_l\|, \quad \forall l \in \{i_1, i_2, \dots, i_k\}$ .
- (d)  $\|f_{i_1, i_2, \dots, i_k} - f(p_l)\| \leq \lambda_{i_1, i_2, \dots, i_k}\|x - p_l\|, \quad \forall l \in \{1, \dots, N\}$

By applying Proposition 18, we obtain  $f_{i_1, i_2, \dots, i_k} = K(f, S)(x)$  and  $\lambda_{i_1, i_2, \dots, i_k} = \lambda(f, S)(x)$ .

**Remark 20.** By applying theorem 19, this method terminates when  $k = l + 1 \leq m + 1$ , where  $m$  is dimension of  $\mathbb{R}^m$ .

**Remark 21.** In step 3, when we solve  $f_{i_1, i_2, \dots, i_k}$  by considering the equation

$$\Gamma(f_{i_1, i_2, \dots, i_k}, f(p_{i_1}), f(p_{i_2}), \dots, f(p_{i_k})) = 0,$$

by replacing  $\|f_{i_1, i_2, \dots, i_k} - f(p_l)\|$  by  $\lambda_{i_1, i_2, \dots, i_k} \|x - p_l\|$ , for  $l \in \{i_1, i_2, \dots, i_k\}$ , this equation is equivalent to

$$a(x)\lambda_{i_1, i_2, \dots, i_k}^4 + b(x)\lambda_{i_1, i_2, \dots, i_k}^2 + c(x) = 0, \quad (32)$$

where  $a(x), b(x), c(x)$  are function only depending on  $x$  and initial data  $x_l, f(p_l)$  for  $l \in \{i_1, \dots, i_k\}$ . The polynomial  $a(x)\lambda_{i_1, i_2, \dots, i_k}^4 + b(x)\lambda_{i_1, i_2, \dots, i_k}^2 + c(x)$ , in fact, is 2-degree polynomial with variable  $\lambda = \lambda_{i_1, i_2, \dots, i_k}^2$ . Therefore, we can solve Equation (32) very fast to obtain exactly the value of  $\lambda_{i_1, i_2, \dots, i_k}$ .

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